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Exactly solvable models through the empty interval method, for more-than-two-site interactions

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Abstract

Single-species reaction–diffusion systems on a one-dimensional lattice are considered, in which more than two neighbouring sites interact. Constraints on the interaction rates are obtained, that guarantee the closedness of the time evolution equation for $E_n(t)$, the probability that n consecutive sites are empty at time t. The general method of solving the time evolution equation is discussed. As an example, a system with next-nearest-neighbour interaction is studied.

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1. Introduction

In contrast to equilibrium systems, which are best analysed using standard equilibrium statistical mechanics, there is no general approach to study systems far from equilibrium. People are motivated to study non-equilibrium systems in one dimension, since these are in principle easier. Different methods have been used to study stochastic models in one dimension, including analytical and asymptotic methods, mean-field methods, and large-scale numerical methods. Some models solved using these methods are studied for example, in [1–11].

There is no universal meaning for the term exactly solvable. For example, in [12–14], solvability means that the evolution equation of n-point functions contains only n- or less-point functions. In [15, 16], solvability means that the S-matrix of the N-particle system is factorized into products of two-particle S-matrices. This means that the S-matrices should satisfy the Yang–Baxter equation. Another meaning of integrability is that the time evolution equation for $E_n(t)$, the probability that n consecutive sites are empty at time t, is closed, that is it can

be expressed in terms of other $E_m(t)$. This method of solving the integrable models is called the empty interval method (EIM).

The empty interval method has been used to analyse the one-dimensional dynamics of diffusion-limited coalescence [17–20]. Using this method, the functions $E_n(t)$ have been calculated. For the cases of finite reaction rates, some approximate solutions have been obtained. EIM has also been generalized to study the kinetics of the q-state one-dimensional Potts model in the zero-temperature limit [21].

In [22], all one-dimensional reaction–diffusion models with nearest-neighbour interactions, exactly solvable through EIM, have been studied. In [23], EIM has also been used to study a specific model with next-nearest-neighbour interaction. In [24], the conventional EIM has been extended to a more generalized form. Using this extended version, a model not solvable by conventional EIM has been studied.

There are few exact results on systems with more-than-two-site interactions, even fewer than exact results for two-site interactions. It seems that reactions in nature are dominated by two-body interactions. There are, however, more-than-two-body interactions in nature, for which a more-than-two-site interaction may be a better approximation than a two-site interaction. Examples are three-body interactions in nuclear physics, and some reactions catalysed by inhomogeneous catalysts.

In [23], a single-species system has been studied, in which diffusion, coagulation, and the three-site production $A\emptyset A \to AAA$ is present, and the rates of diffusion and coagulation are the same. An exact solution has been obtained, from which it is seen that the system exhibits no phase transition. In [25], the same model, but with different coagulation and diffusion rates, has been studied. For this latter system, no exact result has been found. Using the cluster mean field approximation, and Monte Carlo simulations, it has been shown that the system exhibits a phase transition. In [14], systems with more-than-two-site interactions are considered, in which the evolution equation for the one-point function is closed. A classification for these systems is obtained. In the present paper, we consider systems with more-than-two-site interactions, for which a class of n-point functions can be exactly obtained.

The scheme of the present paper is as follows. We consider the most general systems with k-site interactions. Some constraints are imposed on the interaction rates, so that the time evolution equation for $E_n(t)$ is closed. The general method of solving the time evolution equation is also discussed. Finally, as an example, a system with next-nearest-neighbour interactions has been considered in more detail.

2. Models solvable through the empty interval method

Consider a general one-species reaction–diffusion model on a one-dimensional periodic lattice with $L + 1$ sites, with a k-neighbouring-site interaction. We want to find criteria on the interaction rates, that guarantee the solvability of the system via EIM, that is, the closedness of the evolution equation for the probability that n consecutive sites are empty, E_n.

Suppose that the initial condition of the system is translationally invariant. Any configuration of k neighbouring sites is denoted by $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, where $a_i = \circ$ or $\bullet$. $\circ$ ($\bullet$) is used to denote an empty (occupied) site. The rate of transition from a configuration $\mathbf{a}$ to $\mathbf{b}$ is denoted by $\lambda_{\mathbf{a}}^{\mathbf{b}}$. Similar to [22], the interactions with k empty sites as initial or final configuration are not considered here. In other words for any $\mathbf{a}$,

$$\lambda_{\mathbf{0}}^{\mathbf{a}} = \lambda_{\mathbf{a}}^{\mathbf{0}} = 0. \tag{1}$$

Excluding these interactions from the $2^k(2^k - 1)$ possible interactions, $(2^k - 1)(2^k - 2)$ interactions remain to be considered. We want to impose restrictions on $\lambda_{\mathbf{a}}^{\mathbf{b}}$ in such a way that

the evolution equation for $E_n(t)$ is closed. As we will see, the form of evolution equation generally will be different for $n \geqslant k-1$ and $n < k-1$, and also will be different for $n+k > L+2$ and $n+k \leqslant L+2$. So we will treat each case separately.

2.1. The cases $n \geqslant k-1$ and $n+k \leqslant L+2$

To obtain the evolution equation for $E_n(t)$, one should first recognize the source and sink terms. There are two cases. In the first case, the intersection of the empty block and the interacting block is on the left-hand side of the empty block. In the other case, this intersection is on the right-hand side of the empty block. For the first case, the source terms come from

$$a'_1 \cdots a'_l \overbrace{c_1 \cdots c_{k-l} \circ \cdots \circ}^{n} \to b_1 \cdots b_l \overbrace{\circ \cdots \circ}^{n} \tag{2}$$

where $\mathbf{c} \neq \mathbf{0}$. Here $\mathbf{0}$ stands for a block of adjacent empty sites. One also has $l \leqslant k-1$. $\lambda^{\mathbf{0}}_{\mathbf{a}} = 0$ leads to $l \geqslant 1$. So the left source for E_n is

$$S_L = \sum_{l=1}^{k-1} \sum_{\substack{\mathbf{a}',\mathbf{b} \\ \mathbf{c}\neq\mathbf{0}}} \lambda^{\mathbf{b0}}_{\mathbf{a}'\mathbf{c}} P(\mathbf{a}'\mathbf{c}\overbrace{\circ \cdots \circ}^{n-k+l}). \tag{3}$$

Now consider the expansion

$$\sum_{\substack{\mathbf{a}' \\ \mathbf{c}\neq\mathbf{0}}} \lambda^{\mathbf{f}}_{\mathbf{a}',\mathbf{c}} = \sum_{l'=l}^{k-1} \sum_{\mathbf{a}} \lambda^{\mathbf{f}}_{\mathbf{a}\bullet\mathbf{0}} \tag{4}$$

in which $\mathbf{a}$ is an l'-dimensional vector and $\mathbf{0}$ is $(k-l'-1)$-dimensional. So,

$$S_L = \sum_{l=1}^{k-1} \sum_{l'=l}^{k-1} \sum_{\mathbf{a},\mathbf{b}} \lambda^{\mathbf{b0}}_{\mathbf{a}\bullet\mathbf{0}} P(\mathbf{a} \bullet \overbrace{\circ \cdots \circ}^{n+l-l'-1}). \tag{5}$$

If

$$\Lambda^L_{ll'} := \sum_{\mathbf{b}} \lambda^{b_1 \cdots b_l \mathbf{0}}_{a_1 \cdots a_{l'} \bullet \mathbf{0}} \qquad 1 \leqslant l \leqslant k-1 \quad l \leqslant l' \leqslant k-1 \tag{6}$$

is independent of $\mathbf{a}$, then one can sum up $P(\mathbf{a} \bullet \overbrace{\circ \cdots \circ}^{n})$ on the index $\mathbf{a}$. Then

$$S_L = \sum_{l=1}^{k-1} \sum_{l'=l}^{k-1} \Lambda^L_{ll'} P(\bullet \overbrace{\circ \cdots \circ}^{n+l-l'-1}) = \sum_{l=1}^{k-1} \sum_{l'=l}^{k-1} \Lambda^L_{ll'} (E_{n+l-l'-1} - E_{n+l-l'}). \tag{7}$$

One can do similar calculations for the case that the intersection of the interaction block and the empty block is on the right-hand side of the empty block. Defining

$$\Lambda^R_{ll'} := \sum_{\mathbf{b}} \lambda^{\mathbf{0} b_1 \cdots b_l}_{\mathbf{0} \bullet a_1 \cdots a_{l'}} \qquad 1 \leqslant l \leqslant k-1 \quad l \leqslant l' \leqslant k-1 \tag{8}$$

and assuming that it is independent of $\mathbf{a}$, the source term for this case is

$$S_R = \sum_{l=1}^{k-1} \sum_{l'=l}^{k-1} \Lambda^R_{ll'} P(\overbrace{\circ \cdots \circ}^{n+l-l'-1} \bullet). \tag{9}$$

Putting these together, the source term is

$$S = \sum_{l=1}^{k-1} \sum_{l'=l}^{k-1} \left(\Lambda^L_{ll'} + \Lambda^R_{ll'}\right) (E_{n+l-l'-1} - E_{n+l-l'}). \tag{10}$$

Now, let us consider the sink terms. Again we will treat interactions of left- and right-hand sides separately. First consider the left ones. The interactions which contribute to sink terms come from

$$a'_1 \cdots a'_l \overbrace{\circ \cdots \circ}^{n} \to b_1 \cdots b_l c_1 \cdots c_{k-l} \overbrace{\circ \cdots \circ}^{n-k+l} \tag{11}$$

where $\mathbf{c} \neq \mathbf{0}$. The sink term from the left interactions is

$$R_L = -\sum_{l=1}^{k-1} \sum_{\substack{\mathbf{a}',\mathbf{b} \\ \mathbf{c}\neq\mathbf{0}}} \lambda^{\mathbf{bc}}_{\mathbf{a}'\mathbf{0}} P(\mathbf{a}' \overbrace{\circ \cdots \circ}^{n}). \tag{12}$$

$\lambda^{\mathbf{a}}_{\mathbf{0}} = 0$ leads to $l \geqslant 1$. One also has $l \leqslant k-1$. $\lambda^{\mathbf{b}}_{\mathbf{a}}$ is the transition rate, so it is defined only for $\mathbf{a} \neq \mathbf{b}$. But one can extend this definition and define the diagonal terms in such a way that

$$\sum_{\mathbf{b},\mathbf{c}} \lambda^{\mathbf{bc}}_{\mathbf{ad}} = 0. \tag{13}$$

Using this, one arrives at the following equation for R_L

$$R_L = \sum_{l=1}^{k-1} \sum_{\mathbf{a}',\mathbf{b}} \lambda^{\mathbf{b0}}_{\mathbf{a}'\mathbf{0}} P(\mathbf{a}' \overbrace{\circ \cdots \circ}^{n}). \tag{14}$$

Noting that $\sum_{\mathbf{a}'} \lambda^{\mathbf{f}}_{a'_1 \cdots a'_l \mathbf{0}} = \sum_{l'=0}^{l-1} \sum_{\mathbf{a}} \lambda^{\mathbf{f}}_{a_1 \cdots a_{l'} \bullet \mathbf{0}}$, the above equation is reduced to

$$R_L = \sum_{l=1}^{k-1} \sum_{l'=0}^{l-1} \Lambda^L_{ll'} P(\bullet \overbrace{\circ \cdots \circ}^{n+l-l'-1}) \tag{15}$$

where

$$\Lambda^L_{ll'} := \sum_{\mathbf{b}} \lambda^{b_1 \cdots b_l \mathbf{0}}_{a_1 \cdots a_{l'} \bullet \mathbf{0}} \qquad 1 \leqslant l \leqslant k-1 \quad 0 \leqslant l' \leqslant l-1 \tag{16}$$

and it is assumed that $\Lambda^L_{ll'}$ is independent of $\mathbf{a}$.

It is seen that the conditions we have obtained for the source and sink terms for the left interactions, equations (6) and (16), are similar, except for the range of l'. Performing similar calculations for the right interactions, all the conditions coming from the source and sink terms can be summarized as this. The following quantities should be independent of $\mathbf{a}$:

$$\Lambda^L_{ll'} := \sum_{\mathbf{b}} \lambda^{b_1 \cdots b_l \mathbf{0}}_{a_1 \cdots a_{l'} \bullet \mathbf{0}} \qquad 1 \leqslant l \leqslant k-1 \quad 0 \leqslant l' \leqslant k-1 \tag{17}$$

$$\Lambda^R_{ll'} := \sum_{\mathbf{b}} \lambda^{\mathbf{0} b_1 \cdots b_l}_{\mathbf{0} \bullet a_1 \cdots a_{l'}} \qquad 1 \leqslant l \leqslant k-1 \quad 0 \leqslant l' \leqslant k-1. \tag{18}$$

Defining $\Lambda_{ll'} := \Lambda^L_{ll'} + \Lambda^R_{ll'}$, the time evolution equation of $E_n(t)$, for $n \geqslant k-1$ and $n+k \leqslant L+2$, takes the following form:

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = \sum_{l=1}^{k-1} \sum_{l'=0}^{k-1} \Lambda_{ll'} (E_{n+l-l'-1} - E_{n+l-l'}). \tag{19}$$

Note that in this equation, E_0 is defined through

$$E_0 := 1. \tag{20}$$

2.2. The cases $n < k-1$ and $n+k \leqslant L+2$

Now, we want to derive the time evolution equation of $E_n(t)$ when $n < k-1$. Two cases may occur. The first one is that the n adjacent sites which we are focused on are among the k interacting sites, and in the second case a block of these sites is outside those k sites. The result for the second case is similar to that of the preceding subsection, $n \geqslant k-1$. For that case, we only quote the results. However, we study the first case in more detail.

The source terms come from

$$a_1{}' \cdots a_p{}' c_1 \ldots c_n e_1{}' \cdots e_q{}' \to b_1 \cdots b_p \overbrace{\circ\circ\cdots\circ}^{n} d_1 \cdots d_q \tag{21}$$

where $\mathbf{c} \neq \mathbf{0}$, $p+q+n=k$, and $p, q \geqslant 1$. Then the source term is

$$S = \sum_{\substack{p,q=1 \\ p+q=k-n}} \sum_{\substack{\mathbf{a}',\mathbf{e}',\mathbf{b},\mathbf{d} \\ \mathbf{c}\neq\mathbf{0}}} \lambda^{\mathbf{b0d}}_{\mathbf{a}'\mathbf{c}\mathbf{e}'} P(\mathbf{a}'\mathbf{c}\mathbf{e}'). \tag{22}$$

Similar to the previous cases, one can rearrange the sum of the rates in the following form:

$$\sum_{\substack{\mathbf{a}',\mathbf{e}' \\ \mathbf{c}\neq\mathbf{0}}} \lambda^{\mathbf{f}}_{\mathbf{a}'\mathbf{c}\mathbf{e}'} = \sum_{n'=0}^{n-1} \sum_{n''=0}^{q-1} \sum_{\mathbf{a},\mathbf{e}} \lambda^{\mathbf{f}}_{\mathbf{a}\bullet\underbrace{\circ\cdots\circ}_{n'+n''}\bullet\mathbf{e}} + \sum_{n'=0}^{n-1} \sum_{\mathbf{a}} \lambda^{\mathbf{f}}_{\mathbf{a}\bullet\mathbf{0}} \tag{23}$$

In the above equation, $\mathbf{a}$ is a $(p+n-n'-1)$-dimensional vector and $\mathbf{e}$ is a $(q-n''-1)$-dimensional vector. Arranging all these together, one arrives at the following equation for the source term:

$$S = \sum_{\substack{p,q=1 \\ p+q=k-n}} \sum_{n'=0}^{n-1} \left[\sum_{n''=0}^{q-1} \sum_{\mathbf{a},\mathbf{e},\mathbf{b},\mathbf{d}} \lambda^{\mathbf{b0d}}_{\mathbf{a}\bullet\mathbf{0}\bullet\mathbf{e}} P(\mathbf{a}\bullet\mathbf{0}\bullet\mathbf{e}) + \sum_{\mathbf{a},\mathbf{b},\mathbf{d}} \lambda^{\mathbf{b0d}}_{\mathbf{a}\bullet\mathbf{0}} P(\mathbf{a}\bullet\mathbf{0}) \right]. \tag{24}$$

Defining

$$\Lambda_{pq,p'q'} := \sum_{\mathbf{b},\mathbf{d}} \lambda^{b_1\cdots b_p \mathbf{0} d_1\cdots d_q}_{a_1\cdots a_{p'}\bullet\mathbf{0}\bullet e_1\cdots e_{q'}} \tag{25}$$

and

$$\Lambda^L_{pq,p'} := \sum_{\mathbf{b},\mathbf{d}} \lambda^{b_1\cdots b_p \mathbf{0} d_1\cdots d_q}_{a_1\cdots a_{p'}\bullet\mathbf{0}} \tag{26}$$

where $p \leqslant p' \leqslant p+n-1$, $0 \leqslant q' \leqslant q-1$, $p+q=k-n$, and $p, q \geqslant 1$. Assuming that $\Lambda_{pq,p''}$ is independent of $\mathbf{a}$ and $\mathbf{e}$ and $\Lambda^L_{pq,p'}$ is independent of $\mathbf{a}$, one can sum up the terms in (24):

$$S = \sum_{\substack{p,q=1 \\ p+q=k-n}} \sum_{n'=0}^{n-1} \left[\sum_{n''=0}^{q-1} \Lambda_{pq,p'q'} (E_{n'+n''} + E_{n'+n''+2} - 2E_{n'+n''+1}) + \Lambda^L_{pq,p'} (E_{q+n'} - E_{q+n'+1}) \right]. \tag{27}$$

The independence of $\Lambda_{pq,p'q'}$ with respect to $\mathbf{a}$ and $\mathbf{e}$, and $\Lambda^L_{pq,p'}$ with respect to $\mathbf{a}$ is sufficient to guarantee that the above source term is expressible in terms of E_n, but is not necessary. For example, in (22) one can decompose the blocks $\mathbf{c}$ and $\mathbf{a}'$ instead of $\mathbf{c}$ and $\mathbf{e}'$, which leads to another set of sufficient conditions on the rates.

Now, let us consider the sink terms for $n < k-1$:

$$a'_1 \cdots a'_p \overbrace{\circ\cdots\circ}^{n} e'_1 \cdots e'_q \to b_1 \cdots b_p c_1 \cdots c_n d_1 \cdots d_q. \tag{28}$$

The above interaction produces a sink term:

$$R = -\sum_{\substack{p,q=1\\ p+q=k-n}} \sum_{\substack{\mathbf{a}',\mathbf{e}',\mathbf{b},\mathbf{d}\\ \mathbf{c}\neq\mathbf{0}}} \lambda^{\mathbf{bcd}}_{\mathbf{a}'\mathbf{0}\mathbf{e}'} P(\mathbf{a}'\mathbf{0}\mathbf{e}') = \sum_{\substack{p,q=1\\ p+q=k-n}} \sum_{\mathbf{a}',\mathbf{e}',\mathbf{b},\mathbf{d}} \lambda^{\mathbf{b0d}}_{\mathbf{a}'\mathbf{0}\mathbf{e}'} P(\mathbf{a}'\mathbf{0}\mathbf{e}') \tag{29}$$

where (13) has been used in the second equality. Expanding $\sum_{\mathbf{a}',\mathbf{e}'} \lambda^{\mathbf{b0d}}_{\mathbf{a}'\mathbf{0}\mathbf{e}'}$, R can be written in the form

$$R = \sum_{\substack{p,q=1\\ p+q=k-n}} \Bigg[\sum_{q'=0}^{q-1}\sum_{p'=0}^{p-1} \Lambda_{pq,p'q'}(E_{k-p'-q'-2} + E_{k-p'-q'} - 2E_{k-p'-q'-1}) + \sum_{q'=0}^{q-1} \Lambda^R_{pq,q'}(E_{k-q'-1} - E_{k-q'}) + \sum_{p'=0}^{p-1} \Lambda^L_{pq,p'}(E_{k-p'-1} - E_{k-p'}) \Bigg] \tag{30}$$

where we have used definitions (25) and (26) for $\Lambda_{pq,p'q'}$ and $\Lambda^L_{pq,p'}$ but with an extension of the range of p' to $0 \leqslant p' \leqslant p+n-1$. It has also been assumed that $\Lambda_{pq,p'q'}$ is independent of $\mathbf{a}$ and $\mathbf{e}$, and $\Lambda^L_{pq,p'}$ is independent of $\mathbf{e}$. $\Lambda^R_{pq,q'}$ is defined through

$$\Lambda^R_{pq,q'} := \sum_{\mathbf{b},\mathbf{d}} \lambda^{b_1\cdots b_p \mathbf{0} d_1\cdots d_q}_{\mathbf{0}\bullet e_1\cdots e_{q'}} \qquad p+q = k-n \quad 0 \leqslant q' \leqslant q-1 \tag{31}$$

and it is assumed that it is independent of $\mathbf{e}$. Considering (27), (30), and the source- and sink-terms corresponding to the previous subsection, and noting that in these latter terms, one should replace $1 \leqslant l \leqslant k-1$ on the right-hand side of (19) with $k-n \leqslant l \leqslant k-1$, one arrives (for $n < k-1$ and $n+k \leqslant L+2$) at

$$\begin{aligned}\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = {} & \sum_{l=k-n}^{k-1}\sum_{l'=0}^{k-1} \Lambda_{ll'}(E_{n+l-l'-1} - E_{n+l-l'}) + \sum_{\substack{p,q=1\\ p+q=k-n}} \Bigg\{ \sum_{n'=0}^{p-1} \Bigg[\sum_{n''=0}^{q-1} \Lambda_{pq,(p+n-n'-1)(q-n''-1)} \\ & \times (E_{n'+n''} + E_{n'+n''+2} - 2E_{n'+n''+1}) + \Lambda^L_{pq,p+n-n'-1}(E_{q+n'} - E_{q+n'+1}) \Bigg] \\ & + \sum_{n'=0}^{p-1}\sum_{n''=0}^{q-1} \Lambda_{pq,n'n''}(E_{k-n'-n''-2} + E_{k-n'-n''} - 2E_{k-n'-n''-1}) \\ & + \sum_{n'=0}^{p-1} \Lambda^L_{pq,n'}(E_{k-n'-1} - E_{k-n'}) + \sum_{n''=0}^{q-1} \Lambda^R_{pq,n''}(E_{k-n''-1} - E_{k-n''}) \Bigg\}. \end{aligned} \tag{32}$$

2.3. The case $n+k > L+2$

The last case to be considered is the case with $n+k > L+2$. Normally, the case of large L and finite k is of interest, in which one also has $n > k$. We assumed periodic boundary condition for the system. Then the intersection of the k interacting sites and the block of n sites may consist of two disconnected parts, of lengths l and l'. So, one has, in addition to the source terms similar to those of subsection 2.1, a source term coming from

$$\underbrace{a'_1\cdots a'_l \circ \cdots \circ b'_1\cdots b'_{l'}}_{n} c'_1\cdots c'_{k-l-l'} \to \mathbf{0} d_1\cdots d_{k-l-l'}. \tag{33}$$

This leads to a source term

$$S = \sum_{\substack{l,l'=1\\ l+l'=n+k-L-1}} \sum_{\substack{\mathbf{a}',\mathbf{b}',\mathbf{c}',\mathbf{d}\\ \mathbf{a}'\neq\mathbf{0}\text{ or }\mathbf{b}'\neq\mathbf{0}}} \lambda^{\mathbf{0d0}}_{\mathbf{b}'\mathbf{c}'\mathbf{a}'} P(\mathbf{a}' \overbrace{\circ\cdots\circ}^{n-l-l'} \mathbf{b}'\mathbf{c}'). \tag{34}$$

Expanding $\sum_{\substack{\mathbf{a}',\mathbf{b}',\mathbf{c}' \\ \mathbf{a}'\neq\mathbf{0}\text{ or }\mathbf{b}'\neq\mathbf{0}}} \lambda^{\mathbf{0d0}}_{\mathbf{b}'\mathbf{c}'\mathbf{a}'}$, it is seen that if the quantities

$$\Lambda'_{ll',pq} := \sum_{\mathbf{d}} \lambda^{\overbrace{\circ\cdots\circ}^{l'}\,\mathbf{d}\,\overbrace{\circ\cdots\circ}^{l}}_{\underbrace{\circ\cdots\circ}_{q}\bullet\mathbf{c}\bullet\underbrace{\circ\cdots\circ}_{p}} \qquad 0\leqslant p\leqslant l-1 \quad 0\leqslant q\leqslant l'-1$$

$$\Lambda'^{L}_{ll',p} := \sum_{\mathbf{d}} \lambda^{\overbrace{\circ\cdots\circ}^{l'}\,\mathbf{d}\,\overbrace{\circ\cdots\circ}^{l}}_{\underbrace{\circ\cdots\circ}_{l'}\mathbf{c}\bullet\underbrace{\circ\cdots\circ}_{p}} \qquad 0\leqslant p\leqslant l-1 \tag{35}$$

$$\Lambda'^{R}_{ll',q} := \sum_{\mathbf{d}} \lambda^{\overbrace{\circ\cdots\circ}^{l'}\,\mathbf{d}\,\overbrace{\circ\cdots\circ}^{l}}_{\underbrace{\circ\cdots\circ}_{q}\bullet\mathbf{c}\underbrace{\circ\cdots\circ}_{l}} \qquad 0\leqslant q\leqslant l'-1$$

are independent of $\mathbf{c}$, then the source term corresponding to (33) is

$$S = \sum_{\substack{l,l'=1 \\ l+l'=n+k-L-1}} \left\{ \sum_{p=0}^{l-1}\sum_{q=0}^{l'-1} \Lambda'_{ll',pq}(-2E_{n+p+q-l-l'+1} + E_{n+p+q-l-l'} + E_{n+p+q-l-l'+2}) \right.$$
$$\left. + \sum_{p=0}^{l-1} \Lambda'^{L}_{ll',p}(E_{n+p-l} - E_{n+p-l+1}) + \sum_{q=0}^{l'-1} \Lambda'^{R}_{ll',q}(E_{n+q-l'} - E_{n+q-l'+1}) \right\}. \tag{36}$$

Now let us consider the sink terms. Again there are terms similar to those of subsection 2.1, and a new sink term, which is

$$R = -\sum_{\substack{l,l'=1 \\ l+l'=n+k-L-1}} \sum_{\substack{\mathbf{a}',\mathbf{b},\mathbf{c},\mathbf{d} \\ \mathbf{b}\neq\mathbf{0}\text{ or }\mathbf{d}\neq\mathbf{0}}} \lambda^{b_1\cdots b_{l'}\mathbf{c}d_1\cdots d_l}_{\underbrace{\circ\cdots\circ}_{l'}\mathbf{a}'\underbrace{\circ\cdots\circ}_{l}} P(\mathbf{a}'\overbrace{\circ\cdots\circ}^{n}). \tag{37}$$

Using (13), and some appropriate expansion, we find that if

$$\Lambda'^{L}_{ll',p} := \sum_{\mathbf{c}} \lambda^{\overbrace{\circ\cdots\circ}^{l'}\,\mathbf{c}\,\overbrace{\circ\cdots\circ}^{l}}_{\underbrace{\circ\cdots\circ}_{l'}\mathbf{a}\bullet\underbrace{\circ\cdots\circ}_{p}} \qquad l\leqslant p\leqslant k-l'-1 \tag{38}$$

is independent of $\mathbf{a}$, then the above sink term becomes

$$R = \sum_{\substack{l,l'=1 \\ l+l'=n+k-L-1}} \sum_{p=l}^{k-l'-1} \Lambda'^{L}_{ll',p}(E_{n+p-l} - E_{n+p-l+1}). \tag{39}$$

Note that here too, this condition on Λ'^{L} is a sufficient condition for the EIM-solvability of the model. Using (36), (39), and the source- and sink-terms corresponding to those of subsection 2.1, one arrives at

$$\frac{\mathrm{d}E_n}{\mathrm{d}t} = \sum_{l=1}^{L-n-1}\sum_{l'=0}^{k-1} \Lambda_{ll'}(E_{n+l-l'-1} - E_{n+l-l'})$$
$$+ \sum_{\substack{l,l'=1 \\ l+l'=n+k-L-1}} \left[\sum_{p=0}^{l-1}\sum_{q=0}^{l'-1} \Lambda'_{ll',pq}(E_{L-k+p+q+1} + E_{L-k+p+q+3} - 2E_{L-k+p+q+2}) \right.$$
$$\left. + \sum_{p=0}^{k-l'-1} \Lambda'^{L}_{ll',p}(E_{n+p-l} - E_{n+p+1-l}) + \sum_{q=0}^{l'-1} \Lambda'^{R}_{ll',q}(E_{n+q-l'} - E_{n+q+1-l'}) \right] \tag{40}$$

for $n + k > L + 2$ (and $n > k$). Note that the summation limits in the terms corresponding to the source and sink terms coming from the processes investigated in subsection 2.1 have been properly modified.

3. General method of the solution

In the previous section, the evolution equations of E_n were obtained, equations (19), (32) and (40). Investigating (32) and (40), one can see that these equations can be rewritten in the general form of (19), provided one defines E_n for $n < 0$, and $n > L + 1$ properly. Doing this, one arrives at

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = \sum_{l=1}^{k-1}\sum_{l'=0}^{k-1} \Lambda_{ll'}(E_{n+l-l'-1} - E_{n+l-l'}) \tag{41}$$

for any n, with the following constraints (which are actually definitions):

$$\sum_{s=r}^{k-1} M_{rs}(E_s - E_{s+1}) = 0 \qquad -k+2 \leqslant r \leqslant -1 \tag{42}$$

and

$$\sum_{s=L+2-k}^{r} N_{rs}(E_{s-1} - E_s) = 0 \qquad L+2 \leqslant r \leqslant L+k-1. \tag{43}$$

In addition to these, there are two other boundary conditions

$$E_0 = 1 \tag{44}$$

and

$$E_{L+1} = 0. \tag{45}$$

This last condition comes from the fact that if the lattice is initially nonempty, it will never become empty (as is seen from (1)). So, excluding the empty lattice (which remains empty) there will always be at least one particle on the lattice. Equations (42)–(45) are $2k-2$ boundary conditions for the difference equation (41), which is of the same order $2k - 2$. To solve these equations, first consider the stationary solution. This solution $\left(E_n^{\mathrm{P}}\right)$ satisfies

$$\sum_{l=1}^{k-1}\sum_{l'=0}^{k-1} \Lambda_{ll'}\left(E^{\mathrm{P}}_{n+l-l'-1} - E^{\mathrm{P}}_{n+l-l'}\right) = 0 \tag{46}$$

with the same boundary conditions (42)–(45). The solution to (46) is

$$E_n^{\mathrm{P}} = \sum_{p=1}^{2k-2} \alpha_p z_p^n \tag{47}$$

where z_p are the solutions of

$$\sum_{l=1}^{k-1}\sum_{l'=0}^{k-1} \Lambda_{ll'}(z^{l-l'-1} - z^{l-l'}) = 0. \tag{48}$$

This equation has $2k - 2$ roots, one of them is 1. The coefficients α_p can be determined using constraints (42)–(45). The full solution is of the form

$$E_n(t) =: E_n^{\mathrm{P}} + F_n(t) \tag{49}$$

where $F_n(t)$ satisfies an equation similar to (41) but with homogeneous boundary conditions, that is equations (42), (43) and (45) hold for F_n, but (44) is substituted with $F_0 = 0$. To solve the resulting equations, one writes F_n as

$$F_n(t) = \sum_{\epsilon} \mathrm{e}^{\epsilon t} F_{\epsilon,n} \tag{50}$$

where $F_{\epsilon,n}$ satisfies

$$\epsilon F_{\epsilon,n} = \sum_{l=1}^{k-1} \sum_{l'=0}^{k-1} \Lambda_{ll'} (F_{\epsilon,n+l-l'-1} - F_{\epsilon,n+l-l'}) \tag{51}$$

with boundary conditions

$$\begin{aligned} &\sum_{s=r}^{k-1} M_{rs}(F_{\epsilon,s} - F_{\epsilon,s+1}) = 0 \qquad && -k+2 \leqslant r \leqslant -1 \\ &\sum_{s=L+2-k}^{r} N_{rs}(F_{\epsilon,s-1} - F_{\epsilon,s}) = 0 \qquad && L+2 \leqslant r \leqslant L+k-1 \\ &F_{\epsilon,0} = 0 \\ &F_{\epsilon,L+1} = 0. \end{aligned} \tag{52}$$

$F_{\epsilon,n}$ can be written as

$$F_{\epsilon,n} = \sum_{p=1}^{2k-2} \beta_{\epsilon,p} z_{\epsilon,p}^{n} \tag{53}$$

where $z_{\epsilon,p}$ should satisfy

$$\sum_{l=1}^{k-1} \sum_{l'=0}^{k-1} \Lambda_{ll'} (z^{l-l'-1} - z^{l-l'}) = \epsilon. \tag{54}$$

This equation has $2k-2$ roots. The coefficients $\beta_{\epsilon,p}$ can be determined using constraints (52). The condition that there exists a nonzero solution for $\beta_{\epsilon,p}$ is that the determinant of the matrix of coefficients is zero. This is a condition for ϵ. So, in principle, one can solve this equation to obtain the solutions for ϵ, and then the corresponding solution for $z_{\epsilon,p}$. One can then obtain $\beta_{\epsilon,p}$, and $F_n(t)$ is obtained using (53) and (50).

4. A model with three-site interaction

As an example, consider a model with three-site (next-nearest-neighbour) interaction. Denoting the eight possible three-state configurations as

$$\begin{aligned} &\mathbf{0} := (\circ\,\circ\,\circ) \qquad \mathbf{1} := (\circ\,\circ\,\bullet) \qquad \mathbf{2} := (\circ\,\bullet\,\circ) \qquad \mathbf{3} := (\circ\,\bullet\,\bullet) \\ &\mathbf{4} := (\bullet\,\circ\,\circ) \qquad \mathbf{5} := (\bullet\,\circ\,\bullet) \qquad \mathbf{6} := (\bullet\,\bullet\,\circ) \qquad \mathbf{7} := (\bullet\,\bullet\,\bullet) \end{aligned} \tag{55}$$

and the transition rate from the state i to the state j by λ_i^j, one can write the conditions for the solvability of the system through the empty-interval method as

$$\begin{aligned}
&\lambda_6^4 = \lambda_2^4\\
&\lambda_3^1 = \lambda_2^1\\
&\lambda_7^4 = \lambda_5^4 = \lambda_3^4 = \lambda_1^4\\
&\lambda_7^1 = \lambda_6^1 = \lambda_5^1 = \lambda_4^1\\
&\lambda_2^1 + \lambda_2^3 + \lambda_2^5 + \lambda_2^7 = \lambda_6^1 + \lambda_6^3 + \lambda_6^5 + \lambda_6^7\\
&\lambda_3^7 + \lambda_3^6 + \lambda_3^5 + \lambda_3^4 = \lambda_2^7 + \lambda_2^6 + \lambda_2^5 + \lambda_2^4\\
&\lambda_1^2 + \lambda_1^6 = \lambda_3^2 + \lambda_3^6 = \lambda_5^2 + \lambda_5^6 = \lambda_7^2 + \lambda_7^6\\
&\lambda_7^3 + \lambda_7^2 = \lambda_6^3 + \lambda_6^2 = \lambda_5^3 + \lambda_5^2 = \lambda_4^3 + \lambda_4^2\\
&\lambda_7^5 + \lambda_7^1 = \lambda_3^5 + \lambda_3^1\\
&\lambda_6^5 + \lambda_6^1 = \lambda_2^5 + \lambda_2^1\\
&\lambda_7^2 = \lambda_5^2\\
&\lambda_3^2 = \lambda_1^2\\
&\lambda_6^2 = \lambda_4^2.
\end{aligned} \tag{56}$$

For example, independence of Λ_{11}^L with respect to $\mathbf{a}$ gives $\lambda_6^4 = \lambda_2^4$. One of course has also

$$\lambda_0^i = \lambda_i^0 = 0. \tag{57}$$

This is nothing but (1). Using (19) for $1 = k - 2 < n < L - k + 3 = L$, we have

$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = \sum_{l=1}^{2}\sum_{l'=0}^{2} \Lambda_{ll'}(E_{n+l-l'-1} - E_{n+l-l'}) \qquad 1 < n < L. \tag{58}$$

The time-evolution equations for E_1 and E_L come from (32) and (40), respectively:

$$\begin{aligned}
\frac{\mathrm{d}E_1(t)}{\mathrm{d}t} = &\sum_{l'=0}^{2} \Lambda_{2l'}(E_{2-l'} - E_{3-l'}) + \Lambda_{11,10}(E_0 + E_2 - 2E_1) + \Lambda_{11,1}^L(E_1 - E_2)\\
&+ \Lambda_{11,00}(E_1 + E_3 - 2E_2) + \Lambda_{11,0}^L(E_2 - E_3) + \Lambda_{11,0}^R(E_2 - E_3)
\end{aligned} \tag{59}$$

and

$$\begin{aligned}
\frac{\mathrm{d}E_L(t)}{\mathrm{d}t} = &\sum_{l'=0}^{2} \Lambda_{1l'}(E_{L-l'} - E_{L+1-l'}) + \Lambda'_{11,00}(E_{L-2} + E_L - 2E_{L-1}) + \Lambda'^L_{11,0}(E_{L-1} - E_L)\\
&+ \Lambda'^L_{11,1}(E_L - E_{L+1}) + \Lambda'^R_{11,0}(E_{L-1} - E_L).
\end{aligned} \tag{60}$$

These two equations can be rewritten in the general form of (58), provided one adds the boundary conditions corresponding to (42) and (43). These are in fact definitions of E_{-1} and E_{L+2}:

$$\begin{aligned}
\sum_{l'=0}^{2} \Lambda_{1l'}(E_{1-l'} - E_{2-l'}) = &\Lambda_{11,10}(E_0 + E_2 - 2E_1) + \Lambda_{11,1}^L(E_1 - E_2)\\
&+ \Lambda_{11,00}(E_1 + E_3 - 2E_2) + \Lambda_{11,0}^L(E_2 - E_3) + \Lambda_{11,0}^R(E_2 - E_3)
\end{aligned} \tag{61}$$

and

$$\begin{aligned}
\sum_{l'=0}^{2} \Lambda_{2l'}(E_{L+1-l'} - E_{L+2-l'}) = &\Lambda'_{11,00}(E_{L-2} + E_L - 2E_{L-1}) + \Lambda'^L_{11,0}(E_{L-1} - E_L)\\
&+ \Lambda'^L_{11,1}(E_L - E_{L+1}) + \Lambda'^R_{11,0}(E_{L-1} - E_L).
\end{aligned} \tag{62}$$

Equations (58), (61) and (62) can be solved using the general method of the previous section.

Now consider a special case

$$\lambda_7^j = \lambda_6^4 = 0. \tag{63}$$

Conditions (56), and the nonnegativity of the rates, lead to

$$\lambda_2^5 = \lambda_6^5 \qquad \lambda_3^7 = \lambda_2^5 + \lambda_2^6 + \lambda_2^7 \qquad \lambda_6^7 = \lambda_2^7 + \lambda_2^3 \tag{64}$$

and that all other λ_i^j are equal to zero. Equation (58) is then reduced to

$$\dot{E}_n = AE_{n+2} + BE_{n+1} - (A+B)E_n \qquad 1 < n < L \tag{65}$$

where

$$A := \lambda_1^5 + \lambda_1^7 + \lambda_4^5 + \lambda_4^7 \qquad B := \lambda_1^3 + \lambda_3^7 + \lambda_4^6 + \lambda_6^5 + \lambda_6^7. \tag{66}$$

Equation (59) becomes

$$\dot{E}_1 = A'E_3 + B'E_2 - (A' + B')E_1 \tag{67}$$

where

$$\begin{aligned} A' &:= \lambda_1^3 + 2\lambda_1^5 + 2\lambda_1^7 + \lambda_4^5 + \lambda_4^6 + 2\lambda_4^7 - \lambda_5^7 \\ B' &:= -\lambda_1^3 - \lambda_1^5 - 2\lambda_1^7 + \lambda_3^7 - \lambda_4^5 - \lambda_4^6 - 2\lambda_4^7 + 2\lambda_5^7 + \lambda_6^7 \end{aligned} \tag{68}$$

and equation (60) becomes

$$\dot{E}_L = B''(E_{L+1} - E_L) \tag{69}$$

where

$$B'' := \lambda_1^3 + \lambda_1^5 + \lambda_1^7 + \lambda_2^3 + \lambda_3^7 + \lambda_4^5 + \lambda_4^6 + \lambda_4^7. \tag{70}$$

This is in fact a degenerate example of the general case considered in the previous section. Note that $E_n = 0\,(1 \leqslant n \leqslant L+1)$ is obviously a solution. This is expected, since the full lattice does not evolve, as $\lambda_7^j = 0$. Noting that $E_{L+1} = 0$, one can solve (69) to obtain E_L. This is found to be

$$E_L(t) = \alpha_L\, \mathrm{e}^{-B''t}. \tag{71}$$

Using this, one can solve the equation for E_{L-1}, to see that it contains two exponentials, $\exp(-B''t)$ and $\exp[-(A+B)t]$. This is provided $B'' \neq A + B$. (Note that in general $B'' \leqslant A + B$. Equality holds iff $\lambda_2^5 = \lambda_2^7 = 0$.) Let us assume $B'' \leqslant A + B$ and proceed. It is not difficult to see that in other E_n there are also terms like $t^l \exp[-(A+B)t]$. One can write

$$E_n(t) = \alpha_n\, \mathrm{e}^{-B''t} + \sum_{l=0}^{L-n-1} \beta_{n,l} t^l\, \mathrm{e}^{-(A+B)t} \qquad 1 < n \leqslant L+1 \tag{72}$$

where

$$\alpha_{L+1} = \beta_{L+1,l} = 0. \tag{73}$$

Putting this in (65), one arrives at

$$A\alpha_{n+2} + B\alpha_{n+1} + (B'' - A - B)\alpha_n = 0 \tag{74}$$

and

$$(l+1)\beta_{n,l+1} = A\beta_{n+2,l} + B\beta_{n+1,l}. \tag{75}$$

The solution to (74) is

$$\alpha_n = \alpha_L \frac{\xi_1^{L+1-n} - \xi_2^{L+1-n}}{\xi_1 - \xi_2} \tag{76}$$

where ξ_i are the roots of the equation

$$(A + B - B'')\xi^2 - B\xi - A = 0 \tag{77}$$

and α_L is arbitrary. The solution to (75) is

$$\beta_{n,l} = \sum_{s=0}^{l} \frac{B^{l-s}}{(l-s)!} \frac{A^s}{s!} \gamma_{n+l+s} \tag{78}$$

where γ_m are arbitrary constants if $1 < m < L$, and zero otherwise.

So far, all E_n except E_1 have been obtained. Using (67), one can also obtain E_1. It is seen that E_1 contains similar terms and a new exponential term $\exp[-(A' + B')t]$. So, in general there are only three time constants in the system (as long as only the empty intervals are concerned). It may occur that two of these time constants, or all of them, are equal. This does not change the general behaviour of the system. Only the degrees of the polynomials multiplied in the exponentials are changed, and the corresponding coefficients can be calculated similarly.

5. Concluding remarks

Single-species one-dimensional reaction–diffusion systems with k-site interactions were considered, and sufficient constraints needed so that the system be solvable, through the empty-interval method, were obtained. The constraints are that some specific linear combinations of the transition rates should vanish. The general method of obtaining the solutions was discussed, and the case $k = 3$ was studied in detail.

To see the connection between our results and some other works on three-site interactions, two specific examples are considered. In [23], a system with the following interactions is studied, and solved through the empty interval method,

$$\begin{array}{ll} A\emptyset \leftrightarrow \emptyset A & \text{with rate } d \\ AA \rightarrow A\emptyset & \text{with rate } d \\ AA \rightarrow \emptyset A & \text{with rate } d \\ A\emptyset A \rightarrow AAA & \text{with rate } 2d\lambda. \end{array} \tag{79}$$

This set of reactions can be written in terms of three-site interactions, the Hamiltonian of which is the following:

$$H = \frac{d}{2} \begin{pmatrix} -4 & 0 & 4\lambda & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & -2-4\lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{80}$$

where our convention is $A := \binom{1}{0}$, and $\emptyset := \binom{0}{1}$. It is easily seen that these reaction rates satisfy (56), and so this system is a special case of the models considered here.

As another example consider the problem studied in [25]. There, a model similar to [23] but with different reaction rates is studied, using the cluster mean field approximation. It can be seen that, for example, $\lambda_6^4 \neq \lambda_2^4$, and so this model cannot be solved using the empty interval method, unless $c/2 = d$ (in the notation of that paper), which is the same model considered in [23].

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